Gradient, Divergence & curl.

<u>Vector Differential Operator (Del)</u>: Denoted by $\vec{\nabla}$, where

$$\vec{\nabla} = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}.$$

It is an operator also known as Nabla which operates on three physical quantities Gradient, Divergence & curl.

<u>Gradient</u>: If $\phi(x,y,z)$ is a defined and differentiable function in a scalar field, then Gradient of ϕ is

$$\vec{\nabla}\phi = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\phi = \frac{\partial \phi}{\partial x}\hat{i} + \frac{\partial \phi}{\partial y}\hat{j} + \frac{\partial \phi}{\partial z}\hat{k}.$$

which is a vector.

Note: The component of $\vec{\nabla}\phi$ in the direction of a unit vector $\hat{a}$ is given by $\vec{\nabla}\phi \cdot \hat{a}$ which is known as directional derivative in the direction of $\hat{a}$ (rate of change of ϕ in the direction of $\hat{a}$).

<u>Divergence</u>: If $\vec{V}$ defines a differentiable vector field in space, that is $\vec{V} = V_1\hat{i} + V_2\hat{j} + V_3\hat{k}$ is defined and differentiable at each point (x,y,z) in this space, then divergence of $\vec{V}$ is $\vec{\nabla}\cdot\vec{V} = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\cdot\left(V_1\hat{i} + V_2\hat{j} + V_3\hat{k}\right)$

$$= \frac{\partial V_1}{\partial x} + \frac{\partial V_2}{\partial y} + \frac{\partial V_3}{\partial z}.$$

Note: If $\vec{\nabla}\cdot\vec{V} = 0$, then $\vec{V}$ is solenoidal.

<u>curl (rotation)</u>: If $\vec{V}(x,y,z) = V_1\hat{i} + V_2\hat{j} + V_3\hat{k}$ is defined and differentiable at each point (x,y,z) in a certain space, then curl of $\vec{V}$ is

$$\vec{\nabla}\times\vec{V} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_1 & V_2 & V_3 \end{vmatrix}$$

<u>Note</u>: If $\vec{\nabla}\times\vec{V} \neq 0$, then $\vec{V}$ is rotational.

$\vec{\nabla}\times\vec{V} = 0$, then $\vec{V}$ is irrotational.

In case of force $\vec{F}$, if $\vec{\nabla}\times\vec{F} = 0$, then $\vec{F}$ is a conservative force field.

gradient

Q.42 P78 If $\phi = 2xz^4 - x^2y$, find $\vec{\nabla}\phi$ and $|\vec{\nabla}\phi|$ at the point $(2,-2,-1)$

A:- $\vec{\nabla}\phi = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(2xz^4 - x^2y)$

$= \hat{i}(2z^4 - 2xy) + \hat{j}(-x^2) + \hat{k}(8xz^3)$

At the point $(2,-2,-1)$;

$\vec{\nabla}\phi = \hat{i}(2.1 - 2.2.-2) + \hat{j}(-2)^2 + \hat{k}(8.2.-1)$

$= 10\hat{i} + 4\hat{j} - 16\hat{k}$ Ans;

$|\vec{\nabla}\phi| = \sqrt{10^2 + 4^2 + (-16)^2} = \sqrt{372} = \sqrt{4}\sqrt{93} = 2\sqrt{93}$ Ans;

Q.43: If $\vec{A} = 2x^2\hat{i} - 3yz\hat{j} + xz^2\hat{k}$ and $\phi = 2z - x^3y$, find $\vec{A}\cdot\vec{\nabla}\phi$ and $\vec{A}\times\vec{\nabla}\phi$ at the point $(1,-1,1)$.

A:- $\vec{\nabla}\phi = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(2z - x^3y)$

$= -3x^2y\hat{i} - x^3\hat{j} + 2\hat{k}$.

$\therefore \vec{A}\cdot\vec{\nabla}\phi = (2x^2\hat{i} - 3yz\hat{j} + xz^2\hat{k})\cdot(-3x^2y\hat{i} - x^3\hat{j} + 2\hat{k})$

$= -6x^4y + 3x^3yz + 2xz^2$

At the point $(1,-1,1)$; $\vec{A}\cdot\vec{\nabla}\phi = -6.1.-1 + 3.1.-1.1 + 2.1.1$

$= 6 - 3 + 2 = 5$ Ans;

Now $\vec{A}\times\vec{\nabla}\phi = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cancel{\frac{\partial}{\partial x}2x^2} & \cancel{\frac{\partial}{\partial y}-3yz} & \cancel{\frac{\partial}{\partial z}} \\ 2x^2 & -3yz & xz^2 \\ -3x^2y & -x^3 & +2 \end{vmatrix}$

~~$= 6xyz$~~

$$= (-6yz + x^4z^2)\hat{i} - (4x^2 + 3x^3yz^2)\hat{j} + (-2x^5 - 9x^2y^2z)\hat{k}$$

At the point $(1,-1,1)$;

$$\vec{A} \times \vec{\nabla}\phi = (6+1)\hat{i} - (4-3)\hat{j} + (-2-9)\hat{k}$$
$$= 7\hat{i} - \hat{j} - 11\hat{k} \quad \text{Ans:}$$

Q-45: Find $\vec{\nabla}|\vec{r}|^3$.

A:- $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$|\vec{r}| = \sqrt{x^2+y^2+z^2}$ $\quad \therefore |\vec{r}|^3 = (x^2+y^2+z^2)^{3/2}$.

Now $\vec{\nabla}|\vec{r}|^3 = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(x^2+y^2+z^2)^{3/2}$ ——— (i)

$$\hat{i}\frac{\partial}{\partial x}(x^2+y^2+z^2)^{3/2} = \hat{i}\cdot\frac{3}{\cancel{2}}(x^2+y^2+z^2)^{\frac{1}{2}}\cdot\cancel{2}x$$
$$= \hat{i}\,3x(x^2+y^2+z^2)^{1/2}$$

Similarly $\hat{j}\frac{\partial}{\partial y}(x^2+y^2+z^2)^{3/2} = \hat{j}\,3y(x^2+y^2+z^2)^{1/2}$

& $\hat{k}\frac{\partial}{\partial z}(x^2+y^2+z^2)^{3/2} = \hat{k}\,3z(x^2+y^2+z^2)^{1/2}$.

From (i);

$$\therefore \vec{\nabla}|\vec{r}|^3 = 3(x^2+y^2+z^2)^{1/2}(x\hat{i} + y\hat{j} + z\hat{k})$$
$$= (3r)\vec{r} \quad \text{Ans:}$$